ELSEVIER

Economics Letters 47 (1995) 317–325

economics letters

Time-additive representations of preferences when consumption grows without bound

Jim Dolmas

Department of Economics, Southern Methodist University, Dallas, TX 75275, USA

Received 3 May 1994; accepted 30 June 1994

Abstract

Koopmans' classic theorem on the representation of intertemporal preference orders by time-additively separable utility functions is inapplicable to economies where consumption streams grow without bound. This paper provides a 'Koopmans-like' theorem for the case of unbounded consumption growth. Feasible consumption streams obey an asymptotic growth condition, growing in the limit no faster than an arbitrary, fixed reference stream.

Keywords: Additive separability; Growth

JEL classification: D90

1. Introduction

This paper extends Koopmans' (1972) classic result on the representation of preference orderings over time by time-additively separable utility functions to the case of unbounded growth in consumption.

Koopmans' result at once extended to an infinite time horizon the earlier finite-dimensional additive representation theorems of Debreu (1960) and Gorman (1968), while at the same time specializing them to the class of stationary, or recursive, preferences. Koopmans' theorem, however, was limited to preferences over bounded consumption streams.[1] The representation theorem below is a 'Koopmans-like' theorem for the case of unbounded consumption streams.

The sort of growth in consumption which the theorem allows is quite arbitrary, but nonetheless structured in the following way: feasible consumption streams may grow no faster,

[1] As Becker and Boyd (1993) note in their survey of recursive utility theory, Koopmans' set of programs 'bounded in utility' is, given his other assumptions, equivalent to a subset of $(l^\infty)^n$.

SSDI 0165-1765(94)00566-4

asymptotically, than a fixed reference stream. Behavior of the reference stream itself *is* arbitrary, modulo the restriction that it be bounded away from zero.[2]

I also provide a simple representation result for more structured environments, when the reference stream grows in a balanced way, and preferences are homothetic.

2. Preliminaries

2.1. Preference orders and utility representations

Let $\succeq$ denote a complete, transitive binary relation on some consumption set E. For two programs X and Y in E, the relationship $X \succeq Y$ is understood to mean that X is weakly preferred to Y. The preference order $\succeq$ is said to have a *utility representation* if there exists a function $U: E \rightarrow R$ such that for all $X, Y \in E$, $X \succeq Y$ if and only if $U(X) \geq U(Y)$.

We seek conditions under which a preference order over an infinite time horizon has a utility representation of the time-additively separable form. By a 'time-additively separable,' or T.A.S., representation of preferences, we mean a utility function of the form

$$\sum_{t=1}^{\infty} \delta^{t-1} u(x_t) ,$$

where x_t denotes consumption in period t. The function u is variously referred to as the 'felicity function', 'momentary utility function' or 'subutility function'; the parameter $\delta > 0$ is the discount factor, with $\delta < 1$ reflecting a preference for advanced timing of consumption. A T.A.S. utility function is thus completely described by the pair (u, δ).

Obviously, a T.A.S. representation will only be possible when the consumption set E can be expressed as a subset of a countable product of identical 'factor spaces'. The next subsection considers the sort of consumption sets we will employ.

2.2. The consumption set

Denote by s^n the space of all sequences of real n-vectors – i.e. the Cartesian product of infinitely many copies of R^n. If $X \in s^n$, then $X = (x_1, x_2, x_3, \ldots)$, where $x_t = (x_{1t}, x_{2t}, \ldots, x_{nt}) \in R^n$ for all t. With few exceptions, sequences in s^n will be denoted by upper-case letters, and their components by lower-case letters. The number x_{it} denotes an amount of the ith good available for consumption in the tth period. For X, $Y \in s^n$, say that $X \geq Y$ if and only if $x_t \geq y_t$ for all $t = 1, 2, \ldots$, where $x_t \geq y_t$ has the standard interpretation for vectors in R^n. We also define the relation $X \gg Y$ if and only if $x_{it} > y_{it}$ for every i and t. A sequence X is *strictly positive* if $X \gg 0$.

Let ω be an element of s^n. Consider the set

$$A(\omega) = \{X \in s^n : \exists \lambda \geq 0 \text{ with } |X| \leq \lambda |\omega|\} .$$

[2] Since *decaying* economies pose no great problems for additive representations, I do not think that this is a serious drawback.

$A(\omega)$ is the set of programs in s^n which asymptotically grow no faster than ω.[3] For $\omega \gg 0$, define the ω-weighted norm of a vector $X \in s^n$ by

$$\|X\|_\omega = \sup_{i,t} \{|x_{it}|/\omega_{it}\} .$$

$A(\omega)$ is then precisely the set of programs in s^n satisfying $\|X\|_\omega < \infty$. When equipped with this norm, $A(\omega)$ is isometrically isomorphic to $(l^\infty)^n$ – the space of bounded sequences of real n-vectors – endowed with the sup-norm. The isometry here is ϕ: $A(\omega) \to (l^\infty)^n$ defined by $(\phi(X))_{it} = x_{it}/\omega_{it}$.

The consumption sets for the two theorems in this paper are both taken to be the non-negative orthant $A(\omega)^+$ of some $A(\omega)$ generated by an ω which is bounded away from zero by a strictly positive constant program.[4] In Theorem 1 we further assume that ω displays growth at a constant rate. In the main theorem, Theorem 2, all we assume is boundedness away from zero, as described. Modulo this restriction, ω may behave arbitrarily, both over time and across commodities. In particular, it may grow without bound. The restriction that ω be bounded away from zero is needed to guarantee the existence of nontrivial constant programs. In this respect, it is actually stronger than necessary; it is enough that for some good i, lim inf$\{\omega_{it}\} > 0$.

Note that when ω is actually equal to a strictly positive constant program – for example, $\omega = (e, e, e, \ldots)$ where $e = (1, 1, \ldots, 1)$ – then $A(\omega)$ is simply $(l^\infty)^n$. Thus, Koopmans' theorem can be viewed as applying to sets $A(\omega)$, where ω is constant and strictly positive.[5]

Specific applications will generally dictate a choice of ω – for example, ω might be the path of pure accumulation in an optimal growth model or the social endowment in an exchange economy.[6] In either case, the representation theorem will apply on every consumption path relevant to the aggregate economy.

For convenience, we will typically denote the consumption sets by E.

2.3. *Some notation*

For $x \in R^n$ and $X \in E$, the notation (x, X) is shorthand for that program in E whose first component is equal to x and whose subsequent components are given by X – i.e. if $X = (x_1, x_2, \ldots)$, then (x, X) denotes the program $Y = (y_1, y_2, \ldots) \in E$, where $y_1 = x$, $y_2 = x_1$, $y_3 = x_2$, and so on. We denote by S and π the *shift* and *projection* operators, respectively. The shift operator acts on an element of E by deleting its first component and shifting all other components 'to the left' one place – i.e. $(SX)_t = x_{t+1}$. The projection simply takes the first component x_1 of a program X into R^n, so that $\pi X = x_1 \in R^n$. The notation S^k denotes the k-fold application of the shift operator – so that $(S^k X)_t = x_{t+k}$ – while π_k denotes the

[3] In other words, $A(\omega)$ is the Riesz ideal in s^n generated by ω. See Aliprantis et al. (1990).

[4] A constant program here is one for which $\omega_t = \omega_s$ for all t and s. It is not necessarily one for which all components – the ω_{it}'s – are identical.

[5] An observation made by Becker and Boyd (1993).

[6] Aliprantis et al. (1990) give examples where ω is the aggregate endowment of an exchange economy, as does Becker (1991).

projection of the first k components of a program onto $(R^n)^k$. If $SX = X$ – in other words, if X is a constant program – S's which would otherwise appear will be suppressed.

3. Assumptions

Koopmans' (1972) representation theorem showed that the following five assumptions on a preference order $\succeq$ guarantee the existence of a T.A.S. representation (u, δ) when the consumption set is $E = (l^{\infty})^n$. Looking ahead, we state them for the general case $E = A(\omega)^+$ for some $\omega \gg 0$:

(A1) *Continuity*. $\succeq$ is ω-norm continuous on E – the sets $\{X \in E : X \succeq Y\}$ and $\{X \in E : Y \succeq X\}$ are ω-norm closed for every $Y \in E$.

(A2) *Stationarity*. For all x, X and X', $(x, X) \succeq (x, X')$ if and only if $X \succeq X'$.

(A3) *Independence*. For all x, x', X and X', $(x, X) \succeq (x', X)$ if and only if $(x, X') \succeq (x'X')$. Also, for all x, x', y, y', X and X', $(x, y, X) \succeq (x', y', X)$ if and only if $(x, y, X') \succeq (x', y', X')$.

(A4) *Sensitivity*. There exists an x and X such that $X \succ (x, SX)$.

(A5) *K-monotonicity*. X, $Y \in E$ and $(\pi_k X, S^k Y) \succeq (\pi_{k-1} X, S^{k-1} Y)$ for $k = 1, 2 \ldots$ implies $X \succeq Y$.[7]

According to standard additive representation theorems a preference order defined on a *separable* consumption set which satisfies the first *four* of these axioms will have an additive utility representation which is unique up to an affine transformation. The stationarity assumption (A2) will further guarantee that the utility function is of the fixed-discount-factor T.A.S. form. The program space E, when endowed with the ω-norm topology, is not separable however, and hence the standard results cannot be applied directly.[8] This is even the case in Koopmans' treatment on $(l^{\infty})^n$ – hence Koopmans' use of K-monotonicity (A5). The statement of the K-monotonicity assumption above is less than transparent. In words, what it says is that if sequentially replacing the components of a program Y with those of a program X leads to a sequence of (weak) improvements, then X itself must be weakly preferred to Y.

One can easily imagine preference orders which satisfy Koopmans' assumptions, hence have time-additive representations, over a restricted domain of bounded sequences, yet fail to be time-additive when unbounded sequences are allowed. As an example, suppose the consump-

[7] Koopmans referred to this property of preferences as 'monotonicity'. Given that I assume a limited form of monotonicity of preferences in the more traditional sense – and for want of a better name – I chose to refer to Koopmans' 'monotonicity' as K-monotonicity, and the 'monotonicity' of assumption (A6) as 'limited monotonicity'.

[8] $A(\omega)^+$ with the ω-norm is not separable for the same reasons l^{∞} with the sup-norm is not separable.

tion set is that subset $A(\omega)^+$ of R_+^∞ generated by an ω of the form $\{\omega_t\}_{t=1}^\infty = \{\alpha^t\}_{t=1}^\infty$, where $\alpha > 1$, and let $X \succeq Y$ if and only if $U(X) \geq U(Y)$, where

$$U(X) = \limsup_t |x_t/\alpha^t| - \exp\left\{-\liminf_T \sum_{t=1}^T \delta^{t-1} x_t\right\},$$

with $\delta \in (0, 1)$, but $\alpha\delta > 1$. Then $\succeq$ is T.A.S. on l^∞ (since the sum in the 'exp$\{\cdot\}$' converges and $\limsup |x_t/\alpha^t| = 0$) but equal to $\limsup |x_t/\alpha^t|$ for any program for which the sum $\Sigma_{t=1}^T \delta^{t-1} x_t$ diverges. Even for sequences in the latter category, $\succeq$ fails to satisfy only one Koopmans assumption, 'sensitivity', (A4). Clearly, then, allowing for unbounded growth is going to require a stronger set of axioms than Koopmans'. To Koopmans' assumptions, we will add the following:

(A6) *Limited monotonicity*. For any $X \in E$, there are $\lambda \geq 0$ and $\tau \geq 0$ with $(\pi_T X, S^T(\lambda\omega)) \succeq X$ for all $T \geq \tau$. For any $X \in E$, there is also a $\hat{\tau} \geq 0$ with $X \succeq (\pi_T X, 0)$, $\forall T \geq \hat{\tau}$.

(A7) *Impatience*. For each $X \in E$ there is an $\alpha > 0$, a constant program Z and an integer K such that $(\alpha\pi_k X, Z) \succeq X$ for all $k \geq K$.

Limited monotonicity (A6), as its name suggests, is a weakened form of monotonicity in the traditional, order-related sense. If preferences *were* actually monotone – so that $X \geq Y$ implied $X \succeq Y$ – then, by definition of E, there would be for each X a $\lambda \geq 0$ with $(\pi_T X, S^T(\lambda\omega)) \succeq X$ for *every* $T \geq 0$. Likewise, Y would *always* be at least as good as $(\pi_T Y, 0)$ for any T.

The 'impatience' assumption (A7) is a further step than K-monotonicity towards limiting the sensitivity of the preference order to the tail behavior of consumption streams. It states that for any program X, there is a factor α by which it can be scaled, a date K after which it can be truncated, and a constant sequence Z appended in place of its original tail – all in such a way as to leave a program which is weakly preferred to the original.

4. Results

For completeness, we record the following result, which shows that in more structured settings – when growth of ω is balanced and preferences are homothetic – Koopmans' axioms are sufficient to guarantee the existence of a T.A.S. representation:

Theorem 1. Suppose that $\succeq$ satisfies (A1) *through* (A5), *where $\omega_t = \alpha^t \bar{\omega}$, $\forall t$, for some $\bar{\omega} \gg 0$ and $\alpha > 1$. If $\succeq$ is, additionally, homothetic, then $\succeq$ has a T.A.S. representation.*

Proof. Apply the mapping ϕ: $A(\omega)^+ \to (l^\infty)^n$ defined by $(\phi(X))_t = x_t/\alpha^t$ to induce an order on $(l^\infty)^n$ by $X \succeq_\phi Y$ if $\phi^{-1}(X) \succeq \phi^{-1}(Y)$. One shows that $\succeq_\phi$ satisfies (A1) through (A5) on $(l^\infty)^n$ given that $\succeq$ satisfies them on E. Homotheticity is required only to verify stationarity. Now, simply apply Koopmans' theorem to $\succeq_\phi$, yielding a sup-norm continuous T.A.S. utility function U representing $\succeq_\phi$ on $(l^\infty)^n$. This in turn implies a T.A.S. representation for $\succeq$ given by $X \succeq Y$ if and only if $U(\phi(X)) \geq U(\phi(Y))$. □

This result is of limited applicability, though one can conceive of circumstances in which the requirements of the theorem are satisfied. For example, consider an individual who can borrow and lend at a constant interest rate r, but faces an asymptotic non-negativity constraint on borrowing. Feasible consumption streams for such an agent obey $c_t \leq \lambda(1+r)^t$ for some $\lambda \geq 0$ – i.e. the feasible set is a set of the form $A(\omega)^+$ as in the theorem, with $\omega_t = (1+r)^t$.

The main result of the paper is:

Theorem 2. Suppose that $\succeq$ is a preference order satisfying assumptions (A1) *through* (A7), *where ω is bounded away from zero by a strictly positive constant program. Then $\succeq$ has a T.A.S. representation (u, δ).*

Proof. The proof proceeds by steps similar to those in the proof of Koopmans' main proposition, representing $\succeq$ on successively larger subsets of E – first, the subset consisting of sequences whose tails coincide after some finite number of periods, then sequences whose tails are constant. Under the ω-norm, these spaces are equivalent to a subset of finite-dimensional Euclidean space, so standard additive representation theorems apply. At this point we extend the representation to the subset of programs which, after some date T, grow like ω. The final step of the proof shows that this last representation actually represents $\succeq$ on *all* of E. Since the proof is rather lengthy, it is given in a series of lemmas. The first two may be proven exactly as in Koopmans (1972) and hence are simply stated:

Lemma 1. Under assumptions (A1)–(A4), *the restriction of $\succeq$ to the subset of E consisting of programs which agree after some date T has an additive representation $U_T(X) = u(x_1) + \delta u(x_2) + \cdots + \delta^{T-1}u(x_T)(\delta > 0)$ which is unique up to an affine transformation. Furthermore, u and δ are independent of T.*

Lemma 2. If, in addition to (A1)–(A4), *we also assume* (A5), *then the representation from Lemma 1 may be extended to programs which are constant after some date T. Furthermore, $\delta < 1$, and the contribution to total utility of a constant tail $(z,z,z,\ldots)$ is $\delta^T u(z)/(1-\delta)$.*

We then have:

Lemma 3. Under (A1)–(A7), *$\Sigma_{t=1}^{\infty} \delta^{t-1}u(\lambda\omega_t)$ exists for every $\lambda \geq 0$.*

Proof. By 'limited monotonicity' (A6), there is a τ such for all $T \geq \tau$, we have $\lambda\omega \succeq (\pi_T(\lambda\omega),0) \succeq (\pi_\tau(\lambda\omega),0)$. By (A7), there is a constant program Z, an $\alpha > 0$ and a K with $(\alpha\pi_K(\lambda\omega),Z) \succeq \lambda\omega$, hence $(\alpha\pi_K(\lambda\omega),Z) \succeq (\pi_T(\lambda\omega),0)$ for all $T \geq \tau$. Now, 0 and $(\pi_T(\lambda\omega),0)$ are both constant after T, as are $(\alpha\pi_K(\lambda\omega),Z)$ and $(\pi_\tau(\lambda\omega),0)$, for all $T \geq \max\{K,\tau\}$. Applying the utility representation from Lemma 2, we obtain

$$\sum_{t=1}^{K} \delta^{t-1} u(\alpha\lambda\omega_t) + \frac{\delta^K u(z)}{1-\delta} \geq \sum_{t=1}^{T} \delta^{t-1} u(\lambda\omega_t) + \frac{\delta^T u(0)}{1-\delta} \geq \sum_{t=1}^{\tau} \delta^{t-1} u(\lambda\omega_t) + \frac{\delta^\tau u(0)}{1-\delta} .$$

This must hold for every $T \geq \max\{K, \tau\}$. Taking the limit as T goes to infinity, we have

$$\sum_{t=1}^{K} \delta^{t-1} u(\alpha\lambda\omega_t) + \frac{\delta^K u(z)}{1-\delta} \geq \sum_{t=1}^{\infty} \delta^{t-1} u(\lambda\omega_t) \geq \sum_{t=1}^{\tau} \delta^{t-1} u(\lambda\omega_t) + \frac{\delta^\tau u(0)}{1-\delta} .$$

It follows that $\Sigma_{t=1}^{\infty} u(\lambda\omega_t)$ exists for every $\lambda \geq 0$. □

Now, consider $\{X \in E : S^T X = S^T(\lambda\omega)$ for some $\lambda \geq 0\}$. Denote this space, which is still of finite dimension, by E_T. The same considerations that yield Lemmas 1 and 2 imply that the restriction of $\succeq$ to E_T has an additive representation identical to those of the previous lemmas with the exception of the term corresponding to the $T+1$st factor space. Since the elements of this factor space are completely described by their λ's, we may write the utility representation as

$$U_T(X) = u(x_1) + \delta u(x_2) + \cdots + \delta^{T-1} u(x_T) + g_T(\lambda) .$$

This representation is again unique up to an affine transformation. Consideration of this representation for fixed λ implies that it must be an affine transformation of the previous utility representing the restriction of $\succeq$ to the space of ultimately identical programs. Now, consider the utility representation of programs which grow like ω after period $T+1$:

$$U_{T+1}(X) = u(x_1) + \delta u(x_2) + \cdots + \delta^{T} u(x_{T+1}) + g_{T+1}(\lambda) .$$

Application of U_{T+1} to programs in E_T implies, given the uniqueness of both representations, that g_T and g_{T+1} are related by

$$g_T(\lambda) = \delta^T u(\lambda\omega_{T+1}) + g_{T+1}(\lambda) .$$

Substituting recursively, we obtain

$$g_T(\lambda) = \sum_{t=T+1}^{\infty} \delta^{t-1} u(\lambda\omega_t) + \lim_{T\to\infty} g_{T+1}(\lambda) .$$

That the sum on the right-hand side exists has been shown above. As for the limit of $g_{T+1}(\lambda)$, note that $|g_T(\lambda) - g_{T+1}(\lambda)| = \delta^T u(\lambda\omega_T) \to 0$ for any $\lambda \geq 0$. Thus, for any $\lambda \geq 0$, $\{g_T(\lambda)\}_{T=1}^{\infty}$ is a Cauchy sequence of real numbers, and thus has a limit, say $g(\lambda)$. We thus have for any T that $\succeq$ is represented on E_T by

$$U_T(X) = \sum_{t=1}^{T} \delta^{t-1} u(x_t) + \sum_{t=T+1}^{\infty} \delta^{t-1} u(\lambda\omega_t) + g(\lambda) .$$

A careful consideration of stationarity requirements, however, shows that $g(\lambda)$ must be independent of λ, hence may be set to zero.[9]

For any $X \in E$, denote by $U(X)$ the sum $\Sigma_{t=1}^{\infty} \delta^{t-1} u(x_t)$. Arguments similar to those applied to $\lambda\omega$ show that $U(X)$ exists for all $X \in E$. The final step in the proof of the theorem is to show that this U actually represents $\succeq$.

So, let $X, Y \in E$ with $X \succeq Y$, but suppose that $U(Y) > U(X)$. Let $\Delta = U(Y) - U(X) > 0$. As in the proof of Koopmans' (1972) main theorem, consider two comparison paths $Y^T = (\pi_T Y, 0)$ and $X^T = (\pi_T X, S^T(\lambda\omega))$ where λ is as in the first part of the 'limited monotonicity' assumption (A6) – that is, λ is such that $X^T \succeq X$ for T larger than some $\hat{\tau}$. We then have

$$U(X^T) - U(X) = \sum_{t=T+1}^{\infty} \delta^{t-1}[u(\lambda\omega_t) - u(x_t)] ,$$

which can be made less than $\Delta/3$ by taking T sufficiently large. Similarly,

$$U(Y) - U(Y^T) = \sum_{t=T+1}^{\infty} \delta^{t-1} u(y_t) ,$$

which is also smaller than $\Delta/3$ for large enough T.

Combining the expressions $U(Y) - U(X) = \Delta, U(X^T) - U(X) \le \Delta/3$ and $U(Y) - U(Y^T) \le \Delta/3$ we have

$$U(Y^T) - U(X^T) \ge \Delta/3 > 0$$

for all T greater than some T^*. Since X^T and Y^T are elements of E_T, and U represents $\succeq$ on E_T, we have $Y^T \succ X^T$ for all $T \ge T^*$. But, given 'limited monotonicity', we have $Y \succeq Y^T$ and $X^T \succeq X$ for all sufficiently large T. Combining these, we have $Y \succ X$, a contradiction. So, $X \succeq Y$ implies $U(X) \ge U(Y)$.

Now, suppose that $U(X) \ge U(Y)$, but $Y \succ X$. Let $X^T = (\pi_T X, 0)$ and $Y^T = (\pi_T Y, S^T(\lambda\omega))$, where now λ is such that $(\pi_T Y, S^T(\lambda\omega)) \succeq Y$ for large enough T. For such a T, we have $Y^T \succeq Y \succ X \succeq X^T$. As before, the constructed paths are both elements of E_T, and, since U represents $\succeq$ on E_T, we have $U(Y^T) > U(X^T)$ for all T. Let $0 < \Delta \le U(Y^T) - U(X^T)$. Now, by taking T sufficiently large, we can make $U(Y^T) - U(Y) \le \Delta/3$ and $U(X) - U(X^T) \le \Delta/3$. Putting these expressions together, we have $U(X) - U(Y) \ge \Delta/3 > 0$, a contradiction. So, $U(X) \ge U(Y)$ implies $X \succeq Y$. This completes the proof of the theorem. □

One can show that if $\succeq$ were 'myopic', or order continuous, it would automatically satisfy the impatience assumption (A7) as well as K-monotonicity (A5) and 'limited monotonicity' (A6).[10] Loosely, making any change in the tail of a program, if the change is delayed sufficiently, yields a program arbitrarily 'close in order' to the original; order continuity of $\succeq$ then says that programs 'close in order' are 'close in preference'. Assumptions (A5)–(A7) are

[9] To see this, use U_T and U_{T+1} to compare programs of the form $(y, x_2, x_3, \ldots, x_T, x_{T+1}, \lambda\omega_{T+2}, \ldots)$ where y is fixed, applying stationarity and the affine uniqueness of the utility representations.

[10] See Aliprantis et al. (1990) for the definition of order continuity.

all particular instances of this idea. In fact, an order continuous preference would also be ω-norm continuous (A1), since norm convergence implies order convergence.[11] Thus, sensitivity, independence, stationarity and order continuity are also sufficient to guarantee the existence of a T.A.S. representation.

Acknowledgements

This paper is taken from chapter one of my doctoral thesis at the University of Rochester. I wish to thank my advisor, John H. Boyd III, as well as participants in the Theory Workshop at the University of Rochester, for many helpful suggestions. The comments of Tapan Mitra, Mukul Majumdar and other participants at the 1990 Buffalo–Cornell–Rochester theory conference, as well as the comments of Robert Becker and Larry Epstein are also gratefully acknowledged. Any errors, of course, are my own.

References

Aliprantis, C., D. Brown and O. Burkinshaw, 1990, Existence and optimality of competitive equilibrium (Springer-Verlag, New York).

Becker, R.A., 1991, An example of the Peleg and Yaari economy, Economic Theory 1, 200–204.

Becker, R.A. and J.H. Boyd III, 1993, Recursive utility: Discrete time theory, Hitotsubashi Journal of Economics 34, 49–98.

Debreu, G., 1960, Topological methods in cardinal utility theory, in: K.J. Arrow, S. Karlin and P. Suppes, eds., Mathematical methods in the social sciences, 1959 (Stanford University Press, Stanford, CA).

Gorman, W.M., 1968, The structure of utility functions, Review of Economic Studies 35, 387–398.

Koopmans, T.C., 1972, Representation of preference orderings over time, in: R. Radner and C.B. McGuire, eds., Decision and organization: A volume in honor of Jacob Marschak (University of Minnesota Press, Minneapolis).

[11] See Becker and Boyd (1993).