Almost orthogonal outcomes under probabilistic voting: A cautionary example

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Abstract

Probabilistic voting is often invoked in applications where the issue space is multidimensional, with little or nor justification for the form taken by voters' non-policy preferences. I illustrate by way of example the extreme fragility of probabilistic voting equilibria with respect to assumptions about the non-policy elements of voters' preferences. I also offer intuition for the fragility using the social welfare functions which also describe the equilibria.

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1 Introduction

In probabilistic voting, the non-policy terms in voters' preferences—the random disturbances from the viewpoint of the candidates—can be mod-

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eled as additive or multiplicative.[1] At least in 'macro' political economy models, one sees little justification offered for a particular choice between these alternatives.[2] How different can equilibrium voting outcomes be, depending on this choice? This note illustrates, by way of example, that the choice of additive or multiplicative disturbances can lead to outcomes that are not just a bit different—we may, in fact, obtain equilibrium policy vectors that are nearly orthogonal to one another.

The example is a simple calibrated model of redistributive transfers financed with taxes on consumption, capital income, and labor income. In that sense it is a descendant of one the earliest applications of probabilistic voting, Lindbeck and Weibull (1987). The model also bears some similarities to Profeta (2007), a more recent application of probabilistic voting.[3] To put the difference in outcomes into some empirical context, the equilibrium tax vectors are further apart than those of, for example, the US and Denmark, and the sizes of the resulting "welfare states"—transfers as a share of national income—are zero and 34 percent under the alternative assumptions.

Since certain fragilities of probabilistic voting have been pointed out before (Ball, 1999; Slutsky, 1986), it is worth emphasizing what this note is *not* about. It is not about non-existence of equilibria (equilibria exist), nor is it about non-uniqueness (equilibria are unique, given the structure of voter preferences). The caution it suggests is not for the theorist, but rather for the practitioner who, seeking to model voting outcomes over a multidimensional issue space, is considering an "off the shelf" version of probabilistic voting as a solution.[4]

[1]In the language of Banks and Duggan (2005), these are the 'utility difference' and 'utility ratio' models.

[2]See, for example, Yang (1995), Hassler et al. (2005), Profeta (2007), or Alesina et al. (2013). In the same vein, some papers appeal to probabilistic voting to justify maximization of a utilitarian social welfare function, without noting that probabilistic voting—depending on the form of voters' non-policy preferences—can also give rise to very different social welfare functions, as we discuss in section 4 below. See, for example Calahorrano and Lorz (2011) or Kunze (2014).

[3]In the model here and in Profeta (2007), individuals value consumption and leisure, and are taxed to finance redistributive transfers. In Profeta, a multi-dimensional issue space arises because income tax rates are individual-type-specific; here, taxes are paid on consumption and two types of factor income, and agents differ in their factor endowments.

[4]Probabilistic voting has long been used as a solution concept in models where the

In the next section, I present the model that gives rise to the disparate political equilibria—a simple one-period economy with taxes on labor income, capital income and consumption. Tax revenue is used to finance some exogenous government consumption and a lump-sum redistribution to the economy's agents, who differ in the productivity of their labor effort and in their shares in the economy's capital income. The political equilibria are obtained under probabilistic voting, with either additive or multiplicative disturbances (described in greater detail below).

Section 3 describes the calibration and solution of the model and the almost orthogonal outcomes we obtain under additive and multiplicative disturbances. Finally, section 4 offers some intuition for the disparate equilibria, based on the relationship between probabilistic voting outcomes and the maximization of social welfare functions. Additive or multiplicative disturbances correspond to different social welfare functions (Banks and Duggan, 2005); in models of redistribution, the different objectives tilt optima either toward or away from the economy's less affluent members. An appendix gives greater detail on the calibration and numerical solution of the model.

2 A simple model of redistributive taxation

2.1 Economic equilibrium

The model is a simple static model of an aggregate economy, with preferences and technology that are standard in much of macroeconomics. Agents maximize a utility function

$$u(c,n) = c\,(1-n)^{\phi}$$

subject to

$$(1-\tau_N)\,wen + (1-\tau_K)\,s\Pi + T = (1+\tau_C)\,c,$$

where c is consumption, n is labor effort, w is the wage rate, Π is aggregate profits, and T is the lump-sum transfer. The agent's type is a pair (e,s),

dimensionality of the issue space prevents application of the median voter theorem. The conditions for existence of a probabilistic voting equilibrium are typically much less stringent than the conditions sufficient to guarantee a Condorcet winner when the issue space is multi-dimensional (Enelow and Hinich, 1989; Coughlin, 1992).

where e is the type's labor productivity, and s is the type's share of aggregate profits. There is a distribution $f(e,s)$ of types, with $\sum_{(e,s)} f(e,s) = 1$. Let $n(e,s)$ and $c(e,s)$ denote the consumption and work effort of agent type (e,s).

The technology for producing output is Cobb-Douglas, $Y = N^{\alpha}$, where Y is aggregate output and $N \equiv \sum_{(e,s)} f(e,s)n(e,s)e$ is the aggregate effective labor input. Aggregate profits are given by $\Pi = (1-\alpha)\,Y$ and the wage w obeys $w = \alpha Y/N$.

Exogenous government consumption is specified as a fraction g of aggregate output, and generates no utility for individuals. The aggregate resource constraint is thus

$$C = Y - G = (1-g)Y,$$

where $C = \sum_{(e,s)} f(e,s)c(e,s)$.

The government runs a balanced budget—

$$T + gY = \tau_N wN + \tau_K \Pi + \tau_C C$$

—so the issue space can be taken as the set of three-dimensional tax vectors $\tau = (\tau_C, \tau_N, \tau_K)$. The lump-sum transfer can be used only for redistribution, not as a lump-sum tax to finance government consumption $(T \geq 0)$.

For a given tax vector τ, the economy has a competitive equilibrium—prices and quantities such that all agents are maximizing their utility subject to their budget constraints and all markets clear—and that equilibrium yields for each agent an indirect utility over taxes $v_{(e,s)}(\tau)$.

2.2 Political equilibrium

The political environment is standard to models that employ probabilistic voting.

The environment consists of candidates from two parties—A and B—vying for election. Candidates espouse policy platforms—tax vectors, in our case—and are assumed committed to enacting their platforms, if elected. In the probabilistic voting framework, voters' preferences over election outcomes depend on more than just candidates' policy platforms.[5] Formally, the utility that voter $i = (e,s)$ obtains if candidate $j \in \{A, B\}$ wins

[5] For example, voting intentions may depend on voters' perception of the leadership qualities of the candidates, as in models of valence (Schofield, 2004).

(and enacts policy τ) depends on both τ and j. In probabilistic voting we further assume a type of separability—voter i's utility of candidate j enacting τ is a function of j and of i's indirect utility from the tax vector τ, $v_i(\tau)$. The precise form of that separability—additive or multiplicative—will be shown to have a significant impact on the political outcome.

In the additive case, the utility a voter of type i gets from candidate A winning the election and enacting policy τ^A is

$$V_i^A\left(\tau^A\right) = \xi_i^A + v_i\left(\tau^A\right),$$

where $v_i \geq 0$ is the voter's indirect utility function over the choice of tax vector. In a two-party election, a voter of type i votes for candidate A over candidate B if

$$v_i\left(\tau^A\right) - v_i\left(\tau^B\right) > \xi_i^B - \xi_i^A \equiv \psi_i.$$

The ψ_i's—which represent aspects of voters' party preferences apart from the explicit policies—are taken as random from the two candidates' standpoints. Given a distribution of the ψ_i's in the population, assume that each candidate chooses his policy platform to maximize his expected plurality, given the policy choice of the other candidate. A common assumption on $F(\psi_i)$, the CDF of ψ_i, is that it takes logistic form (independent of i).[6] Then, candidate A's expected plurality is

$$2\sum_i f_i \times \left(\frac{\exp\left[v_i\left(\tau^A\right)\right]}{\exp\left[v_i\left(\tau^A\right)\right] + \exp\left[v_i\left(\tau^B\right)\right]}\right) - 1, \tag{1}$$

where f_i is the fraction of type i agents in the electorate.

In contrast to the additive case just described, suppose instead that the utility a voter of type i gets from candidate A winning the election and enacting policy τ^A is

$$V_i^A\left(\tau^A\right) = \exp\left(\xi_i^A\right) v_i\left(\tau^A\right),$$

with $\psi_i \equiv \xi_i^B - \xi_i^A$ still distributed logistically. In this case, A's expected plurality takes the same form as (1) above, but with $\exp\left(v_i\left(\,\cdot\,\right)\right)$ replaced

[6] For example, see Lindbeck and Weibull (1987), Coughlin (1992), and Yang (1995), or the various papers cited in Dow and Endersby (2004).

by $v_i(\cdot)$ —*i.e.*,

$$2\sum_i f_i \times \left(\frac{v_i(\tau^A)}{v_i(\tau^A)+v_i(\tau^B)}\right) - 1. \tag{2}$$

In a probabilistic voting equilibrium, candidates maximize either (1) or (2), as the case may be, taking as given the policy platform of their opponent.

3 Almost orthogonal outcomes

While the model is too simple to be taken as a good description of a real-world economy, I nevertheless try to calibrate it to be roughly consistent with U.S. data. I assume that α, which corresponds to labor's share of national income, is 0.6. I choose ϕ to be consistent with the average agent devoting 30% of his time to work when the tax vector is $\tau_0 = (0.05, 0.25, 0.25)$ (a very rough approximation to the U.S. tax system; see Carey and Rabesona (2002)). Government consumption as a share of output (g) is set equal to 0.15.

For the distribution of agent types, I use two e values and two s values, for a total of four agent types. I calibrate the marginal distribution of e to match the cross-sectional standard deviation of log real wages from Katz and Autor (1999). I calibrate the marginal distribution of s to match the U.S. distribution of wealth by quintiles reported in Budría-Rodríguez et al. (2002). I set the correlation between e and s based on the wealth–earnings correlation reported in Budría-Rodríguez et al. (2002). None of the four types constitutes a majority, of course, though the low-e, low-s type comes close at 47 percent of the population.

The policy decision is a vector of taxes and an associated transfer payment. The issue space is three-dimensional, since a choice of the three tax rates implies a transfer level, via the government's budget constraint. I solve the model numerically on a $101 \times 101 \times 101$ grid of tax vectors in $[0,1] \times [0,1] \times [0,1]$—so τ_C, τ_N, and τ_K each take values ranging from zero to 100 percent in increments of one percentage point. Calculating the competitive equilibrium at each tax vector τ yields indirect utility functions $v_{(e,s)}(\tau)$ for each voter type. I solve for voting equilibria by solving the equivalent social welfare maximizations described in section 4 below.[7]

[7] The calibration and numerical solution are described in greater detail in the appendix

The two very different outcomes we obtain are:

$$\begin{bmatrix} \tau_C \\ \tau_N \\ \tau_K \end{bmatrix} = \begin{bmatrix} 0.00 \\ 0.25 \\ 0.00 \end{bmatrix} \tag{A}$$

which yields a transfer of

$$T = 0$$

and

$$\begin{bmatrix} \hat{\tau}_C \\ \hat{\tau}_N \\ \hat{\tau}_K \end{bmatrix} = \begin{bmatrix} 0.51 \\ 0.03 \\ 1.00 \end{bmatrix} \tag{M}$$

which yields a transfer of

$$\hat{T} = 0.34.$$

Case A is the symmetric equilibrium of the two-candidate, normal form game with payoffs implied by the expected plurality function with additive disturbances (1) from section 2.2. Case M is the equivalent object for candidate payoffs given by (2), the expected plurality function for the multiplicative case. The voting equilibrium in case A leans toward the favorite outcome of the wealthier agents, who prefer zero redistributive transfers and a tax on labor income just sufficient to pay for the exogenous government consumption. The voting equilibrium in case M tilts toward the favorite outcome of the poorest agents, who prefer large transfers financed by taxes on capital and consumption. Section 4 gives the intuition behind these tendencies manifest in the two outcomes.

Obviously, the outcomes are far apart. Using the Euclidean distance between tax vectors as a metric, the tax vectors for the cases (A) and (M) are further apart than those of the most and least generous welfare states among advanced economies.[8]

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sections A.1 and A.2. The MATLAB programs for these computations are available at http://www.jimdolmas.net/economics/current-work.

[8]The corresponding tax vectors for the U.S. and Denmark, for example, are roughly $\tau_{\mathrm{US}} = [0.06, 0.23, 0.27]$ and $\tau_{\mathrm{DK}} = [0.21, 0.40, 0.40]$ (Carey and Rabesona, 2002), for a Euclidean distance of 0.26, or 26 percentage points. The distance between τ and $\hat{\tau}$ above is 1.14, or 114 percentage points.

4 Intuition: Additive disturbances, multiplicative disturbances and social welfare functions

The intuition for the disparate outcomes under additive or multiplicative disturbances can be seen by considering the equivalence between equilibrium outcomes under probabilistic voting and the maximization of a social welfare function. The additive or multiplicative cases correspond to social welfare functions that differ in the weight they place on the utility of the less affluent.

It's easy to verify that the first-order conditions for maximizing (1), evaluated at a symmetric equilibrium ($\tau^A = \tau^B$) are identical to the first-order conditions for maximizing a utilitarian social welfare function:

$$S(\tau) = \sum_i f_i v_i(\tau). \tag{3}$$

Likewise, the first-order conditions for maximizing expected plurality with multiplicative disturbances (2) are (at $\tau_A = \tau_B$) identical to the first-order conditions for maximizing a social welfare function of the form

$$\hat{S}(\tau) = \sum_i f_i \log(v_i(\tau)). \tag{4}$$

These results are essentially Corollary 3 and Corollary 3′ from Banks and Duggan (2005).

Note that in comparison to (3)—in which voter utilities are perfect substitutes—the curvature present in (4) offers greater gains from the transfer of utils from the relatively well-off to the relatively worse-off. As a result, the preferences of poorer agents will receive effectively more weight under (4) than under (3).

To see this, suppose that $\tau^\star$ maximizes the utilitarian social welfare function (3). Then, for τ near $\tau^\star$, (4) can, to a first-order approximation, be written as[9]

$$\hat{S}(\tau) \cong \eta + \sum_i \left(\frac{f_i}{v_i(\tau^\star)}\right) v_i(\tau),$$

which has the same form as the utilitarian social welfare function (3)—up to the constant η—but gives relatively more weight to types with lower values of $v_i(\tau^\star)$.

[9]Using the fact that $\log(v_i(\tau)) \cong \log(v_i(\tau^\star)) + (1/v_i(\tau^\star))(v_i(\tau) - v_i(\tau^\star))$.

If the preferred policy vectors of our model economy's poor and rich are far apart, the probabilistic voting outcomes will be far apart. This is the intuition for the disparate outcomes shown above.

5 Conclusion

Depending on whether voters' policy and non-policy preferences interact as $\xi + v(\tau)$ or $\exp(\xi)v(\tau)$, we can—for the same underlying economy—obtain probabilistic voting equilibria more disparate than the most and least expansive welfare states we observe in the real world. This fact should caution practitioners against simply applying probabilistic voting to determine policy outcomes without careful consideration of—and justification for—the particular form assumed.

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Appendix A Calibration and solution details

This appendix describes in greater detail the calibration and numerical solution of the model.

A.1 Calibrating the model

The technology parameter α governs labor's share of national income; I set this to 0.6, a typical value. Government consumption of goods and services, as a share of output—the parameter g—is set equal to 0.15, also typical for models calibrated to U.S. experience.

To construct the distribution of agent types, I separately construct marginal distributions of s (shares in aggregate profit) and e (labor productivity). Let f_S denote the marginal density of s. I assume $f_S = \{0.6, 0.4\}$ for $s \in \{0.06/0.60, (1-0.06)/0.4\}$, so the bottom 60 percent of the population hold claim to 6 percent of aggregate profits, following data on the U.S. wealth distribution in Budría-Rodríguez et al. (2002), and treating profit in the model as analogous to wealth in the data.

Similarly, let f_E be the marginal density of $e \in \{e_1, e_2\}$. I set $f_E = \{0.6, 0.4\}$ and set e_2/e_1 to give a standard deviation of log real wages equal to 0.55 (Katz and Autor, 1999).[10] Lastly, I normalize e so that $\sum_i f_{E,i} e_i = 1$.

The resulting voter types have $(e_1, e_2) = (0.55, 1.68)$ and $(s_1, s_2) = (0.10, 2.35)$, with identical marginal densities $\{0.6, 0.4\}$. I construct the joint density of the four types as a convex combination of the density that would obtain under zero correlation between e and s and the density that would obtain under perfect correlation. The weight on the density under perfect correlation is 0.47, the correlation between wealth and earnings reported in Budría-Rodríguez et al. (2002). The resulting joint density is given by

$$f = [f(e_i, s_j)]_{i,j=1,2} = \begin{bmatrix} 0.47 & 0.13 \\ 0.13 & 0.27 \end{bmatrix}$$

The model has a single preference parameter, ϕ, which affects the allocation of time spent in market work. I set ϕ so that the average hours worked in the economy, given a tax vector $\tau_0 = (0.05, 0.25, 0.25)$, equal 0.3, a standard value in macroeconomic models.

This is accomplished by calculating equilbrium prices and quantities for a given value of ϕ and adjusting the value of ϕ—via a bisection algorithm—until the model yields

$$\sum_{(e,s)} f(e,s) n(e,s) = 0.3$$

[10]For that calculation, I use the fact that the standard deviation of log real wages can be rewritten as $\sqrt{f_{E,1} f_{E,2}} \log(e_2/e_1)$.

to within 10^{-3}. The corresponding value of ϕ is 1.0715.

A.2 Numerical solution

Economic equilibrium (given taxes) is straightforward to compute. An agent's first-order conditions imply

$$\phi \frac{c(e,s)}{1-n(e,s)} = \frac{(1-\tau_N)we}{1+\tau_C}$$

Aggregating over all agents, and substituting $C = (1-g)Y$, $w = \alpha Y/N$ and our normalization $\sum f(e,s)e = 1$ gives a simple closed-form expression for equilibrium aggregate effective labor:

$$N = \frac{\alpha(1-\tau_N)}{\alpha(1-\tau_N) + \phi(1-g)(1+\tau_C)}.$$

Given N, other equilibrium quantities, prices and individual choices and utilities are readily calculated. It's thus straightforward, given a tax vector τ, to evaluate the social welfare functions described in section 4.

As described in section 3, the equilibrium tax vectors are found by solving the model—and evaluating the social welfare functions (3) and (4)—on a $101 \times 101 \times 101$ grid of tax vectors in $[0,1] \times [0,1] \times [0,1]$. The two vectors reported in section 3 maximize the two social welfare functions over this grid.