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Balanced-growth-consistent recursive utility

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Abstract

Koopmans's 'recursive utility' has proven useful in a number of dynamic modelling contexts. Nonetheless, recursive utility has not made significant inroads into what one would expect to be a natural haven – models of balanced growth, whether 'exogenous' or 'endogenous'. Mainly, this is due to the dearth of interesting recursive utilities which are consistent with balanced growth. In this paper I provide conditions on the aggregator which guarantee the existence of a recursive utility function which is consistent with balanced growth. The result in turn shows how a family of such utility functions may be constructed. I also provide a generalization of Jones and Manuelli's theorem on the existence of optimal endogenous growth.

Key words: Recursive utility; Balanced growth; Endogenous growth
JEL classification: D90; O41

1. Introduction

Koopmans's (1960) generalization of time-additively separable utility to 'recursive utility' held out a great deal of promise for a more economically reasonable treatment of preferences in dynamic models. Recursive utility preserves the most attractive features of additive utility – time consistency and amenability to dynamic programming – while dispensing with the artifice of a fixed rate of impatience.

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A notable problem with recursive utility, however, has been the difficulty in finding interesting recursive utility functions which are consistent with balanced growth – 'consistent' in the sense of having constant marginal rates of substitution along consumption streams with constant growth rates. Thus, while recursive utility has proven its usefulness in a number of contexts,[1] it has to date made no significant inroads into the 'new growth theory' – where balanced growth is generally taken to be an equilibrium or optimal outcome.[2] While growth theory has come a long way from the basic neoclassical model with exogenous technical change, the progress within the field has been limited to the technology side of the problem – the preference side has remained very much the same.

The purpose of this paper is to show that this need not be the case – there *are* nonadditive recursive utility functions which are consistent with balanced growth. What's more, such functions are fairly simple to specify. The logic of the argument is quite simple: If a utility function is homogeneous of some degree γ and recursive, then it is consistent with balanced growth. The problem then becomes one of finding such utility functions – which on its surface is not an easy task. However, I show that the 'aggregator approach' to recursive utility, first developed in Lucas and Stokey (1984), and later refined by Boyd (1990), can be employed to make this task manageable.[3] The end result is a large 'parametric' family of balanced-growth-consistent recursive utility functions. An attractive feature of the family is that a number of existence results from the additive case – in particular, existence of optimal paths in a Ramsey model and existence of endogenously growing optimal paths – carry through with only minor modification. Hypotheses cast in terms of a fixed discount factor need only be recast in terms of upper and lower bounds on the discounting inherent in the recursive utility specification. To demonstrate this point, I provide generalizations of the Brock–Gale condition for existence of optimal paths (Brock and Gale, 1969) and Jones and Manuelli's theorem on the existence of optimal endogenous growth (Jones and Manuelli, 1990).

The next section of the paper develops the theory of balanced-growth-consistent recursive utility, while Section 3 examines the existence and growth of optimal paths in a Ramsey model with balanced-growth-consistent recursive utility. Some proofs, as well more formal aspects of some sections, have been

[1]For example, in the study of optimal growth with heterogenous agents (Lucas and Stokey, 1984; Benhabib et al., 1987), in generating complex dynamics (Benhabib et al., 1990), in analyzing basic questions in international trade (Obstfeld, 1982), in analyzing the effects of capital income taxation (Chamley, 1986), and in resolving empirical puzzles in finance (Epstein and Zin, 1989).

[2]Barro and Sala-i-Martin (1992) contains a nice survey of a number of the 'new growth' models, illuminating some fundamental similarities between the seemingly disparate models they consider.

[3]The 'aggregator', defined formally in the next section, is Koopmans's name for the function which combines current consumption and future utility to yield present utility.

relegated to the Appendix, the specific contents of which are indicated at various points in the text. I offer some concluding comments in Section 4.

2. Balanced-growth-consistent recursive utility

2.1. Preliminaries

Suppose the utility associated with a consumption stream $C = (c_1, c_2, \dots)$ is given by $U(C)$. Let $U_t(C)$ denote the marginal unity of consumption at date t, so that the marginal rate of substitution between consumption in periods t and $t + 1$ is

$$MRS_{t,t+1}(C) = U_t(C)/U_{t+1}(C).$$

If we think about situating these preferences in a model of capital accumulation – and if we wish to have balanced growth as a possible outcome of either equilibrium or optimal choices – then $MRS_{t,t+1}(C)$ must be independent of t whenever consumption grows at a constant rate.[4] The notation here – $MRS_{t,t+1}(C)$ – is meant to emphasize that in the most general case the marginal rate of substitution at any date will depend on the *entire path* of consumption, making constancy along balanced paths of consumption something quite special.

In the additively separable case, with momentary utility u and discount factor δ, the marginal rate of substitution between periods t and $t + 1$ depends only on consumption at dates t and $t + 1$. That is,

$$MRS_{t,t+1}(C) = u'(c_t)/\delta u'(c_{t+1}).$$

Constancy along balanced paths is had by requiring further that this marginal rate of substitution depend only on the *ratio* of c_t to c_{t+1}. This is done by assuming that u is either homogeneous of some degree γ or logarithmically homogeneous.

What about the general recursive case? Let us first be more precise about the meaning of recursivity. A utility function U is *recursive* if there exists a function W defined on the space of one-period consumptions and the range of U such that

$$U(c_1, c_2 \dots) = W(c_1, U(c_2, c_3 \dots))$$

[4]The intertemporal marginal rate of substitution will either be equated to a marginal product of capital – the case of optimal growth – or an interest rate – the case of equilibrium growth. Either quantity, the marginal product or the relative price, must be constant along a balanced path.

for every path of consumption $(c_1, c_2 \ldots)$. In other words, with recursive utility, the utility at $t = 1$ of a consumption stream C is a function of current consumption, c_1, and utility at date $t = 2$. Utility at date $t = 2$ is in turn a function – in particular, the *same* function – of consumption at $t = 2$ and utility at $t = 3$, and so on *ad infinitum.* The function W which combines current consumption and future utility is known as the *aggregator*, an appellation due to Koopmans. In the language used in the study of separable utility functions over finitely many commodities, W would be the *macro function.* In terms of functional structure, a recursive utility function displays *strict separability* between current consumption and future consumption. More than that, a recursive utility function is *stationary.* Stationarity means that if a path C starting today is at least as good as another path C', then the path which gives some c today and C from tomorrow on is at least as good as the path with c today and C' from tomorrow on, for any c – and *vice versa.* Stationarity is only possible with an infinite number of commodities.[5]

Obviously, any time-additive utility function with a fixed discount factor is recursive, for consider:

$$U(c_1, c_2 \ldots) = \sum_{t=1}^{\infty} \delta^{t-1} u(c_t) = u(c_1) + \sum_{t=2}^{\infty} \delta^{t-1} u(c_t) = u(c_1) + \delta U(c_2, c_3 \ldots).$$

Here, the aggregator is given by $W(x, y) = u(x) + \delta y$.

When utility is recursive, but nonadditive, the marginal rate of substitution between consumption in periods t and $t + 1$ will in general depend on more than just consumption at those two dates, but this dependence will be limited to consumption on or after those dates. The marginal rate of substitution between periods t and $t + 1$ will be independent of consumption of dates $t - 1$ or earlier. To see this, note that with recursive utility, the marginal utility of consumption at date t – found by repeated application of the chain rule to $U(C) = W(c_1, W(c_2, W(\cdots)\cdots)$ – is given by

$$\begin{aligned} U_t(c_1, c_2 \ldots) = {} & W_2(c_1, U(c_2, c_3, \ldots)) \times W_2(c_2, U(c_3, c_4, \ldots)) \times \cdots \\ & \times W_2(c_{t-1}, U(c_t, c_{t+1}, \ldots)) \times W_1(c_t, U(c_{t+1}, c_{t+2}, \ldots)), \end{aligned}$$

so that taking the ratio $U_t(C)/U_{t+1}(C)$ yields

$$MRS_{t,t+1}(C) = \frac{W_1(c_t, U(c_{t+1}, \ldots))}{W_2(c_t, U(c_{t+1}, \ldots)) W_1(c_{t+1}, U(c_{t+2}, \ldots))}.$$

[5]The standard reference for the finite-dimensional theory is Blackorby, Primont, and Russell (1978).

Despite its independence from past consumption, this MRS can clearly become quite complicated when utility is nonadditive. Nevertheless, a similar principle applies as in the additive case: homogeneity of some degree γ is sufficient to yield constancy of the MRS along balanced paths of consumption. This is the substance of:

Proposition 1. *If utility is homogeneous of some degree γ and recursive with a once-differentiable aggregator W, then the marginal rate of substitution between consumption in adjacent periods will be constant whenever consumption grows at a constant rate.*

Proof. See Appendix. The basic idea of the proof is that if U is recursive and degree-γ homogeneous, then U's aggregator W satisfies homogeneity of the form $W(\lambda x, \lambda^\gamma y) = \lambda^\gamma W(x, y)$. This has implications for the homogeneity properties of W_1 and W_2, which together with that of U, guarantee constancy of $MRS_{t,t+1}$ along balanced paths. ■

This result – which tells us that certain *sorts* of recursive utility functions are consistent with balanced growth – is not, unfortunately, particularly useful by itself. Specifying a recursive utility function – i.e., writing down a U – is in general not an easy task, and the homogeneity constraint makes it even less so.

Our problem, really, is how to generate nontrivial homogeneous recursive utility functions in as simple a way as possible. The next section shows how the 'aggregator approach' to recursive utility – specifying a W rather than a U – can be adapted to this purpose.

2.2. *The aggregator approach*

Historically, Koopmans's pioneering work on recursive utility took the utility function U – or a preference order representable by a U – as primitive and studied the question of the existence of the aggregator W. In other words, Koopmans asked: What properties of preferences guarantee recursivity? In the practice of dynamic modelling, however, it's typically easier to work directly with the aggregator, taking it, rather than the utility function, as primitive. One might, for example, think more readily about meaningful restrictions on the trade-off between current consumption and future utility, and about impatience – restrictions which can be embodied in the functional specification of an aggregator – rather than restrictions on U.[6]

[6]See, for example, Lucas and Stokey, *op. cit.*, or Benhabib et al., *op. cit.*, for applications which impose conditions directly on W. As Becker and Boyd (1993) note in their survey of recursive utility theory, this approach actually has antecedents in Fisher's *Theory of Interest* and Hayek's *Pure Theory of Capital*.

One potential problem with taking the aggregator as primitive, however, is verifying the existence of an underlying utility function. For an arbitrary W, there typically will be no function U which satisfies the recursion $U(c_1, c_2, \dots) = W(c_1, U(c_2, \dots))$. This problem was first dealt with by Lucas and Stokey (1984), for the case of bounded aggregators. Boyd (1990) then introduced techniques which accommodated unbounded aggregators as well. Both methodologies rely on finding U as the fixed point of a functional mapping under certain restrictions on W.

The Lucas–Stokey–Boyd results render operational the 'aggregator as primitive' approach. One begins with an aggregator that has properties desirable in one's particular modelling context. If that aggregator then meets the conditions of the existence results in either Lucas and Stokey or Boyd, one is guaranteed of the existence of the recursive utility function, even if one cannot obtain an analytical expression for it.[7] Becker and Boyd (1993) present an exhaustive survey of these and other results, many of them being the product of the two authors' own considerable work in this area.

For our purposes, if we can show that there are restrictions on the aggregator which guarantee homogeneity of the recursive utility function – and if those restrictions are not inconsistent with existence of the recursive utility function – our problem will be solved. The result in the next section shows that this is in fact possible.

2.3. Theorem

To set the stage for the main result, we need to understand a bit more of the Lucas–Stokey–Boyd methodology. Taking the aggregator W as primitive, a recursive utility function – if one exists – will be the fixed point of a mapping T_W on an appropriate space of functions defined over streams of consumption. For ξ in this space, $T_W\xi$ is defined by

$$(T_W\xi)(c_1, c_2, \dots) = W(c_1, \xi(c_2, c_3, \dots)).$$

Clearly, if U satisfies $T_W U = U$, we have a recursive utility function.

Both Lucas and Stokey and Boyd give conditions – both on W and on the domain of T_W – under which T_W is a strict contraction, hence has a fixed point. When this is true, the recursive utility function U not only exists, but is also the

[7] In principle, the Lucas–Stokey–Boyd results, because they rely on the Banach fixed point theorem, imply that the resulting U can be approximated to any degree of accuracy. In practice, this is not always possible.

unique limit of the sequence of iterates $T_W^n \xi = T_W(T_W^{n-1}\xi)$ for any initial ξ in the domain of the mapping.[8]

The fact that U is the fixed point of a strict contraction is at the heart of our Theorem. The method of proof, given formally in the Appendix, is standard for such structures: If a contraction mapping preserves a particular property of the functions in its domain, and if that property holds up under the limits being taken, then the fixed point of the mapping must have that property.[9] Here, I show that a particular homogeneity restriction on the aggregator guarantees that T_W maps degree-γ homogeneous functions into degree-γ homogeneous functions. Further, the limit[10] of a sequence of degree-γ homogeneous functions is itself a degree-γ homogeneous function. Hence, the fixed point of the mapping must be a degree-γ homogeneous function.

Theorem 1. Suppose W satisfies the conditions of the Continuous Existence Theorem (Boyd, 1990) and is such that $W(\lambda x, \lambda^\gamma y) = \lambda^\gamma W(x, y)$ for all x and y and all $\lambda > 0$, for some γ. Then, the recursive utility function U exists and is homogeneous of degree γ. If W is also once-differentiable, U is consistent with balanced growth in the sense of Proposition 1.

Proof. See Appendix.

Note that the Theorem doesn't require x to be a scalar; it applies equally to the cases of a single consumption good and many consumption goods. Most importantly, this result implies that a family of balanced-growth-consistent aggregators can be defined by

$$W(x, y) = u(x)w(y/u(x)),$$

where u is homogeneous of degree γ, and w and u together determine an aggregator W which satisfies the conditions of Boyd's theorem. This is the 'parametric' family of utility functions to which I referred in the introduction – the two parameters are u and w.

The reader may have noticed that, since utility here is ordinal, there's nothing particularly special about γ as the degree of homogeneity. We could always render such a utility function homogeneous of degree one, for example, by an

[8]The principal difference between Lucas–Stokey and Boyd is that Boyd, by utilizing weighted contraction techniques, allows a much larger domain for the mapping T_W, one which includes both bounded and unbounded functions.

[9]Lucas (1978), for example, employs this method repeatedly to establish the properties of the value function in his asset pricing model.

[10]In the relevant topology.

appropriate monotone transformation. Given $W(x, y) = u(x)w(y/u(x))$, there's always a degree-one homogeneous $\tilde{u}$ and a $\tilde{w}$ such that $\tilde{W}(x, y) = \tilde{u}(x)\tilde{w}(y/\tilde{u}(x))$ corresponds to the same preferences and is homogeneous of degree one.[11] I keep the degree of homogeneity γ distinct and explicit in order to maintain an analogy with the additive CRRA case, which can be rendered degree-one homogeneous only by sacrificing additivity. The set of feasible pairs (u, w) can be at least partially circumscribed by requirements such as $W_1 > 0$, $W_2 > 0$ and uniformly bounded time perspective. We will consider each of these restrictions – as well as the other conditions of Boyd's existence theorem – in the next section.

The family has at least one familiar member, since the additively separable CRRA utility functions can be had by taking u to be

$$u(c) = c^\gamma/\gamma$$

and w to be

$$w(z) = 1 + \delta z.$$

We'll see shortly that this is not the only member of this family.[12]

What does $MRS_{t,t+1}(C)$ look like for this family of recursive utility functions? Let z_t stand for $u(c_t)^{-1}U(c_{t+1}, \ldots)$, the argument of w at date t. Suppose too that for now $c_t \in R_+$, the case of a single consumption good.[13] A little algebra shows that when $W(x, y) = u(x)w(y/u(x))$, the marginal rate of substitution between periods t and $t + 1$ is given by

$$MRS_{t,t+1}(C) = \frac{u'(c_t)}{u'(c_{t+1})}\frac{w(z_t) - z_t w'(z_t)}{w(z_{t+1}) - z_{t+1}w'(z_{t+1})}\frac{1}{w'(z_t)},$$

which combines some standard and some nonstandard features – where by 'standard' I mean 'familiar from the additive case'. The ratio $u'(c_t)/u'(c_{t+1})$, for

[11] Let $\tilde{u} = u^{1/\gamma}$, and define $\tilde{w}$ by $\tilde{w}(z) = w(z^\gamma)^{1/\gamma}$.

[12] One can see another additive specification by considering $u(c) = c$ and $w(z) = z^\delta$. This doesn't quite fit the conditions of the Theorem, but nonetheless gives rise to a time-additive utility function with logarithmic momentary utility: The recursion $U(c_1, c_2, \ldots) = c_1(U(c_2, \ldots)/c_1)^\delta$ implies $\tilde{U}(c_1, c_2, \ldots) = (1 - \delta)\ln c_1 + \delta\tilde{U}(c_2, \ldots)$, where $\tilde{U} = \ln(U)$.

[13] The case of many consumption goods – in which we'd be concerned with the marginal rate of substitution between $c_{i,t}$ and $c_{i,t+1}$ – is a straightforward extension of what follows. Also, the *intra*temporal marginal rate of substitution between goods i and j at some date t will always be given simply by $u_i(c_t)/u_j(c_t)$, where u_k denotes the kth partial derivative of u.

example, is standard in this sense. When u is given by $u(c) = c^\gamma/\gamma$, this is simply $(c_{t+1}/c_t)^{1-\gamma}$. The term $1/w'$ is standard, too, though less obviously – it's just the discounting, which in the additive case is constant and equal to $1/\delta$. The term in $w - zw'$ is harder to interpret, but reflects the fact that current consumption really has two effects on utility – it generates a direct gain (captured by the w) and also affects discounting of future utility (captured in the zw' term).

Now, consider a balanced path with $c_{t+1}/c_t = \theta$ for all t. Since U and u are both degree-γ homogeneous, $U(c_{t+2}, c_{t+3}, \dots) = \theta^\gamma U(c_{t+1}, c_{t+2}, \dots)$ and $u(c_{t+1}) = \theta^\gamma u(c_t)$. This implies that $z_{t+1} = z_t = z_1$ along a balanced path. Further, u' is homogeneous of degree $\gamma - 1$. The MRS thus reduces to

$$MRS_{t,t+1}(C) = \theta^{1-\gamma}/w'(z_1),$$

which is independent of t.

Imagine now that preferences are situated in a Ramsey model of capital accumulation. We have $z_1 = U(c_2, c_3, \dots)/u(c_1) = \theta^\gamma U(c_1, c_2, \dots)/u(c_1)$. Along an optimal path, $U(c_1, c_2, \dots) = J(k_0)$, where J is the value function and k_0 is initial capital. One can show that if U is degree-γ homogeneous and the feasible set is a convex cone, then J is degree-γ homogeneous as well.[14] Further, the optimal policy functions – which give the optimal choices of current consumption and next-period's capital given current capital – are homogeneous of degree one. Let $c(k)$ denote the consumption policy. We then have $z_1 = \theta^\gamma J(k_0)/u(c(k_0)) = \theta^\gamma k_0^\gamma J(1)/k_0^\gamma u(c(1)) = \theta^\gamma J(1)/u(c(1))$, which is independent of initial capital.[15] Thus, the rate of balanced growth determined by equating the MRS to a constant marginal product of capital is independent of initial capital.

The theorem can be readily extended to the case where some subset of goods must remain constant along a balanced path – for example, in models in which leisure is a good and the agent has a finite endowment of time in each period. In such cases, an aggregator of the form

$$W(c, l, y) = u(c, l)w(y/u(c, l))$$

will work. If c and l denote the scalar values of consumption and leisure, the aggregator gives rise to the familiar intratemporal efficiency condition

$$u_2(c, l)/u_1(c, l) = (\textit{real wage})_t.$$

If u is homogeneous of degree γ in c, then consumption and real wages can share a common balanced growth rate, while leisure hours remain constant.[16]

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[14]The proof is analogous to the proof of the Theorem above.

[15]In the multiple goods case, z_1 is independent of the *scale* of initial capital: letting $\kappa = (1/\|k_0\|)k_0$, we have $z_1 = \theta^\gamma J(\kappa)/u(c(\kappa))$.

[16]A formal proof for the more general case is given in Dolmas (1994).

2.4. Restrictions on w and u

Feasible choices of the functions w and u are constrained by both economic and technical considerations. From an economic standpoint, we want the resulting aggregator W to be such that both current consumption and future utility are desirable – that is, that both W_1 and W_2 are positive.[17] The technical considerations address the existence of a utility function U given an aggregator W. The most basic of these is a condition referred to in Boyd (1990) as 'uniformly bounded time perspective'. This condition places a uniform bound on the rate at which the aggregator discounts future utility.[18] Since I've been implicitly assuming that W_2 exists, what uniformly bounded time perspective amounts to is that $\bar{\delta} \equiv \sup_{x,y} W_2(x, y)$ be finite, guaranteeing that future utility is discounted by at least $\bar{\delta}$.[19] The constant $\bar{\delta}$ plays a prominent role in the contraction-mapping arguments which derive the utility function U from the aggregator W. In the additively separable case, this bound is simply given by the fixed discount factor.

We may consider what these restrictions imply for the functions u and w. To avoid notational clutter, let's continue to let z stand for the argument of w, so $z = y/u(x)$. Then, assuming w is differentiable, differentiation reveals that

$$W_1(x, y) = u'(x) \times \{w(z) - zw'(z)\}.$$

Thus, $W_1 > 0$ requires that $u'(x)$ and $w(z) - zw'(z)$ be either both positive or both negative. W_2 is a bit simpler, since $W_2(x, y)$ reduces to $w'(z)$. Thus, $W_2 > 0$ if and only if w is strictly increasing.

Given that W_2 is simply w', uniformly bounded time perspective requires that w be such that $\sup\{w'(z)\}$ is finite. If w is concave, this rules out Inada-type conditions at zero, while if w is convex, this seems to require that w be asymptotically affine or linear.

A number of 'technical' conditions remain to be considered, completing the 'conditions of the continuous existence theorem', which guarantee the existence of the recursive utility function. I show in the Appendix that, for this particular class of aggregators, the remaining existence conditions boil down to requiring that W be continuous and that $\beta^{\gamma}\bar{\delta}$ be less than one, where β is the maximal one-period growth rate of consumption and $\bar{\delta} = \sup_z w'(z)$. Note that this is simply an analogue of the familiar Brock–Gale condition for existence of

[17]In fact, the desirability of current consumption is not necessary for the existence of a recursive utility function. See Boyd (1990).

[18]Loosely speaking, an *upper* bound on the amount of *patience*.

[19]When W_2 does not exist, a Lipschitz condition on W with respect to its second argument also suffices. See Boyd (1990).

optimal paths in the additive case – we have simply replaced the fixed discount factor δ with the upper bound of the variable 'discount factors' implied by the aggregator.[20]

2.5. Examples

Since u must be homogeneous of degree γ, the real choice here lies in selecting a function w. This section presents some examples of possible functions w, all of which can be parametrized, together with u, so as to satisfy the restrictions outlined in the last section. Since Boyd's existence restrictions require that preferences be over consumption streams in a particular space, we need to be clear about what that space is. In all cases, I will assume that this space is the collection of all consumption streams which are dominated by some multiple of $B = (1, \beta, \beta^2, \dots)$, where $\beta \geq 1$. We can always take $\beta = 1$ to recover the case of uniformly bounded consumption, but since our interest is in balanced growth, we'll want to allow for $\beta > 1$ as well.[21]

Consider first the following specification of w:

$$w(z) = \frac{1}{\rho}[1 + \bar{\delta} z]^{\rho},$$

where I assume $\rho \in (0, 1)$. There is clearly no transformation of U which will yield an additive representation. Let us consider how the restrictions outlined in the previous section come into play in this case. It's probably easiest to begin by looking at w'. For this specification, we have

$$w'(z) = \bar{\delta}[1 + \bar{\delta} z]^{\rho - 1} \quad \text{or} \quad w'(z) = \rho \frac{w(z)}{z} \frac{\bar{\delta} z}{1 + \bar{\delta} z}.$$

With $\rho \in (0, 1)$, w is concave – so $w(z) > zw'(z)$ for all z. Since W_1 is just $u' \times \{w(z) - zw'(z)\}$, $W_1 > 0$ is guaranteed if $u' > 0$. Further, w' is decreasing, with $\sup_{z \geq 0} w'(z) = w'(0) = \bar{\delta}$, so uniformly bounded time perspective is handled quite neatly in this specification. The other existence conditions[22] again simply boil down to requiring $\bar{\delta}\beta^{\gamma} < 1$.

Note that the '1' which accounts for the lion's share of the cumbersomeness of these expressions is actually quite essential. In general, some constant term like

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[20]As we'll see in Section 3.1, $\beta^{\gamma}\bar{\delta} < 1$, together with the usual continuity and compactness conditions, also suffices to guarantee the existence of optimal paths in the one-good Ramsey model.

[21]I'm thinking here of the case of a single consumption good. For multiple consumption goods, assume that $B = (b_1, b_2, \dots)$, where $b_t = \beta^{t-1}(1, 1, \dots 1)$.

[22]'Contractiveness' and 'ϕ-boundedness', as described in the Appendix.

the '1' must be present in this sort of specification, since we must bound the slope of w – either at the origin or asymptotically, depending on whether w is concave or convex.

Note, too, that while this example illustrates preferences which are *consistent* with balanced growth, it has not been selected to guarantee that growth will be *chosen* by an agent with these preferences in any particular context. The next section shows that if an optimum exists in the one-sector Ramsey model, the optimal paths of consumption and capital will grow without bound whenever $\inf_k f'(k) > 1/\inf_C W_2(c_1, U(c_2, c_3, \dots))$. The example above fails to satisfy this condition regardless of f, since $\inf W_2 = 0$. The following w will generate an aggregator with both $\sup W_2 < +\infty$ and $\inf W_2 > 0$:

$$w(z) = \underline{\delta} z + \frac{1}{\rho}[1 + (\bar{\delta} - \underline{\delta})z]^{\rho}.$$

As the notation suggests, $\underline{\delta}$ is the infimum of $w'(z)$. The supremum, just as in the example immediately above, is again $\bar{\delta}$.

Another example in this vein is

$$w(z) = \alpha + \underline{\delta} z + (\bar{\delta} - \underline{\delta})\ln(1 + z).$$

The implied aggregator has the property that the variable 'discount factor', W_2, is a convex combination of $\bar{\delta}$ and $\underline{\delta}$, since $w'(z) = \underline{\delta} + (\bar{\delta} - \underline{\delta})/(1 + z)$, or

$$w'(z) = \frac{z}{1+z}\underline{\delta} + \frac{1}{1+z}\bar{\delta}.$$

3. Optimal capital accumulation

The last section has demonstrated that balanced-growth-consistent recursive utility functions are simple to construct. This section will demonstrate that they are fairly simple to work with. Situating these preferences in a Ramsey model of optimal capital accumulation, I consider two basic questions: When will optimal paths exist? and When will optimal paths growth without bound? Our answer to the latter question will be a generalization of Jones and Manuelli's theorem on the existence of optimal endogenous growth (Jones and Manuelli, 1990).

3.1. Existence of optimal paths

In the one-sector Ramsey model with additive utility, existence of optimal paths is guaranteed if, in addition to continuity of the momentary utility function u and production function f, we have also $\beta^\gamma \delta < 1$, where β is the slope of an affine upper bound on f, γ is the asymptotic exponent of u, and δ is the fixed discount factor.[23] This condition is typically referred to as the Brock–Gale condition for the existence of optimal paths (Brock and Gale, 1969). We'll see that an analogous condition holds for nonadditive recursive utility of the balanced-growth-consistent variety, the only difference being a replacement of δ by $\bar{\delta}$, the upper bound on the variable discount factor inherent in the aggregator.

The model is described by a production function f and an aggregator W. Assume that f is continuous and nondecreasing, with $f(0) \geq 0$. Assume also that f has an affine upper bound with slope β. Assume that W is of the form $W(x, y) = u(x)w(y/u(x))$ with u continuous and homogeneous of degree γ, and $w(0)$ nonzero. The Ramsey problem is to maximize $U(c_1, c_2, \dots) = W(c_1, U(c_2, \dots))$ subject to $c_t + k_t \leq f(k_{t-1})$, $c_t \geq 0$ and $k_t \geq 0$ for all t, given k_0.

The existence result for this model is:

Proposition 2. If, in the model described above, we have $\beta^\gamma \bar{\delta} < 1$, then an optimal path exists from any initial stock.

Proof. The method of proof is indirect, relying on the fact that existence, uniqueness, and continuity of the value function are sufficient to guarantee that optimal paths exist. To that end, we will use the following result, which is proved by Becker and Boyd (1993):

Lemma. Suppose $W(\cdot, 0)$ is continuous on R_+ and there is an increasing continuous function μ: $R_+ \to R_{++}$ with $\sup_k\{W(f(k), 0)/\mu(k)\}$ finite and $\sup_k\{\mu(f(k))/\mu(k)\} < 1/\bar{\delta}$. Then, the Bellman equation has a unique continuous solution.

Since continuity of u guarantees that $W(x, 0) = u(x)w(0)$ is continuous, our work will be done if we can demonstrate that $\beta^\gamma \bar{\delta} < 1$ implies the existence of a function μ satisfying the conditions of the Lemma. Since $W(f(k), 0) = u[f(k)]w(0) = f(k)^\gamma u(1)w(0)$, if μ grows at least like $f(k)^\gamma$, then $\sup\{W(f(k), 0)/\mu(k)\}$ will be finite. This, and positivity of μ, will be assured if μ takes the form

$$\mu(k) = 1 + (\beta k)^\gamma,$$

[23] An affine upper bound of a function f is simply an affine function g with $g(k) \geq f(k)$ for all k. Any proper concave function has an affine upper bound. The function u has γ as an asymptotic exponent if $u(c)$ eventually grows like c^γ. When u is actually homogeneous of degree γ, the degree of homogeneity and the asymptotic exponent coincide.

where, recall, β is the slope of an affine upper bound on f. With this μ, and again given that f is dominated by an affine function with slope β, $\sup\{\mu(f(k))/\mu(k)\}$ will be less than β^{γ}. Thus, $\beta^{\gamma}\bar{\delta} < 1$ is sufficient to yield $\sup\{\mu(f(k))/\mu(k)\} < 1/\bar{\delta}$, and thus sufficient to guarantee the result in the Lemma. ■

The role of the function μ in the Lemma is to eliminate the standard dynamic programming assumption of bounded one-period returns. Maximal one-period returns – $W\ (f(k), 0)$ – rather than being bounded, may grow no faster than μ, which in turn grows no faster than $\bar{\delta}^{-1}$ along feasible paths.

Proposition 3 can be readily generalized to a multi-sector model – specifically, a model with m consumption goods and n capital goods produced according to a production correspondence, call it Φ, rather than a production function. Continuity and compact-valuedness of Φ replace continuity of f. The other assumptions – modulo obvious changes to accommodate the new number of goods – remain the same. In the hypotheses of the Lemma concerning the function μ, one simply replaces $W(f(k), 0)$ with $\max\{W(c, 0)$: c feasible from $k\}$ and $\mu(f(k))$ with $\max\{\mu(k')$: k' feasible from $k\}$. If a μ exists which satisfies the modified hypotheses, then optimal paths exist from any initial k. Since this result may be obtained from existing results in the literature and, moreover, would take us well beyond the intent of the present section – which is simply to illustrate the ease of application of balanced-growth-consistent recursive utilities – I will not present the proof here.[24]

3.2. *A growth theorem*

I will conclude with a generalization of Jones and Manuelli's (1990) theorem to the case of recursive utility. The purpose of this section is illustrative and is not meant to imply that balanced-growth-consistent recursive utilities are limited in their applicability to models of this form. They may replace time-additive utilities in any infinite-horizon endogenous growth model, whether a model of 'equilibrium' or 'optimal' growth.

As in the proof of Jones and Manuelli's theorem, I work with a set of necessary conditions for optimality and show that, under a joint assumption on preferences and technology, the marginal utility of current consumption goes to

[24]The general case may be viewed alternatively as a simple generalization of Boyd's (1990) result for the one-sector case or as a modification of Sorger's result for a multi-sector reduced form model (Sorger, 1992, Lemma 2.9). What distinguishes Sorger's result from our result, as well as from Boyd's, is that Sorger assumes the 'one-period return' $W(c, 0)$ is bounded by a function which contains infoirmation only about the maximal growth rate of consumption (Sorger, 1992, Definition 2.3). The result here, and in Boyd, allows the function which bounds the one-period return to depend *both* on maximal consumption growth *and* the aggregator W. Hence, Sorger's key condition for existence is $\beta\bar{\delta} < 1$, as compared with our $\beta^{\gamma}\bar{\delta} < 1$.

zero along any optimal path from strictly positive initial stocks of capital. The method of proof is different here, however, in that the necessary conditions I look at are profit maximization conditions rather than Euler equations.[25] A benefit of this approach is that it allows us to consider more general technologies than are allowed within the Jones–Manuelli framework – for example, multi-sector models, models with adjustment costs, or models with nondifferentiable technologies, such as Leontief models. That said, I will first prove the result for Jones and Manuelli's technology, at which point it should be clear how one could incorporate more general technologies.

The model which Jones and Manuelli work with is a type of one-sector Ramsey model with multiple capital goods and a single consumption–investment good. The simplest way to describe the model is probably formally. If $k \in R^n_+$ is current capital, then current output is $f(k)$ where f has the usual properties of concavity, continuity and differentiability. Output is divided between consumption, c, and investment in next period's capital, i. Capital goods are perfect substitutes on the supply side, in the sense that next period's capital, k', and investment are related by $i = \sum_{i=1}^{n} k'_i$. To keep the notation simple, I have abstracted from the depreciation of capital; incorporating it would yield no great insights while considerably complicating the exposition.

For now, let preferences be described by a recursive utility function U with an associated aggregator W. Assume that W is increasing, concave, bounded below, and differentiable on the interior of its domain. Assume that $\bar{\delta} = \sup W_2$ is finite and $\underline{\delta} = \inf W_2$ is positive.

To guarantee growth of the optimal path, suppose that f and $\underline{\delta}$ are such that there is a vector of capital stocks $\hat{k} > 0$ and a $\hat{c} > 0$ with

$$f(\hat{k}) - \underline{\delta}^{-1} \sum_{i=1}^{n} \hat{k}_i \geq \hat{c}.$$

Moreover, suppose that this inequality holds for all nonnegative scalar multiples of $\hat{k}$ and $\hat{c}$. Stated differently, assume that the graph of f contains the ray through a point of the form $(\hat{k}, \hat{c}, \underline{\delta}^{-1}\hat{k})$, with $\hat{k}$ positive and $\hat{c}$ strictly positive.[26] The role of this assumption in guaranteeing growth will presently become clear.

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[25]The conditions will look familiar to students of turnpike theory. The seminal work in this vein is Weitzman (1973).

[26]This assumption is weaker than Jones and Manuelli's and is also the condition for the more general case. Suppose that there are n capital goods and m consumptions goods. Let the technology be described by a production correspondence Φ. We interpret $(c, k') \in \Phi(k)$ to mean that (c, k') is a feasible combination of current consumption and next period's capital, given current capital k. The growth condition in this case reads as follows: There are $\hat{k} > 0$ and $\hat{c} \gg 0$ with $\lambda(\hat{c}, \underline{\delta}^{-1}\hat{k}) \in \Phi(\lambda\hat{k})$ for all $\lambda \geq 0$.

In this convex environment, one can readily show that, if an optimal path exists from strictly positive initial stocks, there exist supporting prices – marginal utilities of consumption and marginal values of capital – such that the optimal path is 'profit-maximizing' at those prices in the following sense: in each period t the value of the one-period 'input–output' combination (k_{t-1}, c_t, k_t), consisting of capital brought into the period, consumption produced within the period, and capital taken out of the period, is maximal over the set of all such feasible combinations.

To be precise, let $\{c_t, k_{t-1}\}_{t=1}^{\infty}$ denote an optimal path from $k_0 \gg 0$, and let J again denote the value function. At any date t, the profit maximization condition is

$$q_t c_t + \delta_t p_t k_t - p_{t-1} k_{t-1} = \max_{k, c, k'} \{q_t c + \delta_t p_t k' - p_{t-1} k\},$$

where the maximization is over all c, k, and $k' \geq 0$ with $c + \sum_{i=1}^{n} k'_i \leq f(k)$. In this expression, q_t is the date-t marginal utility of consumption – i.e., $q_t = W_1(c_t, J(k_t))$, if W_1 exists, or the corresponding supergradient of W.[27] The vector p_t is the date-t marginal value of capital – that is, p_t is a supergradient, or derivative, if one exists, of J at the point k_t. Finally, $\delta_t = W_2(c_t, J(k_t))$.

Since $(\hat{k}, \hat{c}, \underline{\delta}^{-1}\hat{k})$ is a feasible choice for the maximization on the right-hand side of the above expression – and since any scalar multiple of this combination is also a feasible choice – the profit-maximization condition can only hold if $(\hat{k}, \hat{c}, \underline{\delta}^{-1}\hat{k})$ yields a nonpositive profit:

$$0 \geq q_t \hat{c} + \delta_t p_t (\underline{\delta}^{-1}\hat{k}) - p_{t-1}\hat{k}, \quad \forall t.$$

We will show that this last inequality implies that q_t, the marginal utility of current consumption, goes to zero over time. First, note that since W is strictly increasing in current consumption, and $\hat{c}$ is strictly positive, $q_t\hat{c} > 0$. Also, $\delta_t/\underline{\delta} \geq 1$ – since $\underline{\delta}$ has been defined as $\inf W_2$. Combining this information with the last inequality yields $p_{t-1}\hat{k} > p_t\hat{k}$ for all t. Further, $p_t\hat{k} \geq 0$ for all t, since the value function is clearly nondecreasing in capital, and $\hat{k} > 0$. So, the sequence of real numbers $\{p_t\hat{k}\}_{t=0}^{\infty}$ is decreasing and bounded below, hence Cauchy. Next, take this information back to the previous inequality, which can be

[27] If F is a function from R^h, say, to R, then a vector $w \in R^h$ is a *supergradient* of F at the point x in the domain of F if and only if $F(x) - wx \geq F(y) - wy$ for every y in the domain of F. Supergradients generalize the notion of derivatives, and will in many cases exist where derivatives fail to exist, for example at 'kinks' or boundary points.

rewritten as

$$|p_{t-1}\hat{k} - p_t\hat{k}| > q_t\hat{c} > 0, \quad \forall t.$$

It follows that $\lim_t q_t\hat{c} = 0$ or, since $\hat{c} > 0$, $\lim_t q_t = 0$.

This result – that q_t goes to zero along an optimal path – will hold in any model which yields the profit-maximization condition with which we began as a necessary condition for optimality.[28] This will be the case if W is increasing in current consumption, proper, concave, and bounded below; if W_2 exists on the interior of W's domain; and if the graph of the production function or correspondence is convex with nonempty interior.[29] This leads us to the following more general result:

Theorem 2. Consider a multi-sector Ramsey model with technology given by a production correspondence Φ and preferences by a recursive utility function with aggregator W. Assume that Φ and W satisfy the assumptions of the preceding paragraph. If, further, there exists a $\hat{k} > 0$ and $\hat{c} \gg 0$ such that $\lambda(\hat{c}, \underline{\delta}^{-1}\hat{k}) \in \phi(\lambda\hat{k})$ for all $\lambda \geq 0$, then the vector of marginal utilities of current consumption goes to zero along any optimal path from strictly positive initial stocks.

The result implies unbounded growth of consumption in the additive case, if u is strictly increasing and concave. When there are many consumption goods, q_t is equal to $Du(c_t)$, the vector of marginal utilities of current consumption. If u is strictly increasing in each consumption good and concave, then q_t can go to zero only if $\|c_t\|$ goes to infinity. In other words, some subset of the consumption goods is growing without bound.

What about the more general case of recursive utility? Recall that $q_t = W_1(c_t, J(k_t))$. With an arbitrary aggregator W, it's going to be difficult to say what happens to c_t as q_t goes to zero, due to the interaction between current consumption and future utility. But, with aggregators of the form $W(x, y) = u(x)w(y/u(x))$, the question has a simple answer – yes, we do get growth – provided that u is strictly increasing and concave, and $w(z) - zw'(z)$ is bounded away from zero. To see this, recall that $W_1(x, y)$ is given by $u'(x)\{w(z) - zw'(z)\}$, which we have assumed to be positive. Since u is strictly increasing, $u' > 0$, and the assumption about $w(z) - zw'(z)$ implies that there

[28] When there is more than one consumption good, q_t will be a vector.

[29] The proof of this statement parallels the proofs of similar 'supporting price' results in the turnpike literature, and relies on well-known results in convex and nonsmooth analysis. [See, for example, McKenzie (1986) for similar results, Clarke (1983) and Rockafellar (1974) for details of the mathematics. A formal proof of the above statement, and the subsequent theorem and corollary, is given in Dolmas (1994).]

is a $\Delta > 0$ with $u'(x)\{w(z) - zw'(z)\} \geq u'(x)\Delta$. Thus, $q_t \geq u'(c_t)\Delta$, so as q_t approaches zero, $u'(c_t)$ must approach zero as well. Given the concavity and strict monotonicity of u, this can only occur if c_t approaches $+\infty$. As with the additive case, if there are multiple consumption goods, analogous reasoning shows that $\|c_t\|$ approaches $+\infty$.

In sum, we may record the following corollary to Theorem 3, which encompasses both the additive and nonadditive cases:

Corollary 1. If in addition to the assumptions of Theorem 3, we have $W(x, y) = u(x)w(y/u(x))$, *where* u *is strictly increasing and concave, and* $w(z) - zw'(z)$ *is bounded away from zero, then* $\limsup \|c_t\| = +\infty$ *along any optimal path from strictly positive initial stocks.*

3.3. Examples

Some examples might be in order at this point. Following Jones and Manuelli, assume there is now also a single capital good, and let the production function be given by $f(k) = \beta k + g(k)$ for some concave function $g \geq 0$. Assume the utility function is defined by an aggregator of the form $W(x, y) = u(x)w(y/u(x))$, with u homogeneous of degree γ and w such that $\bar{\delta} = \sup w'$ is finite and $\underline{\delta} = \inf w'$ is positive. If, in addition to continuity and so forth, $\beta^{\gamma}\bar{\delta} < 1$, then the recursive utility function exists as does an optimal path. Applying the analysis above, if we also have $\beta\underline{\delta} > 1$, then the optimal path grows without bound.

It's simplest to see the growth condition at work in this example by looking at the balanced case, where g is identically zero, so that the feasible set is a convex cone. Equating the marginal rate of substitution with the marginal product of capital, we obtain

$$\theta^{1-\gamma}/w'(z_1) = \beta,$$

where the notation on the left-hand side of this expression is exactly as in the previous sections – i.e., θ is the growth rate of consumption and $z_1 = J(k_1)/u(c_1)$. We can see clearly that θ is greater than one under our assumptions by noting:

$$\theta^{1-\gamma} = \beta w'(z_1) \geq \beta\underline{\delta} > 1.$$

In this sort of setting, one can also perform rather simply a comparative dynamics exercise showing the dependence of θ on the technology parameter β. Since z_1 is endogenous, we clearly need a second equation, in addition to the

Euler equation. But this is easily obtained from Bellman's equation,

$$J(k_0) = u(c_1)w(J(k_1)/u(c_1)),$$

which can be rearranged to give

$$J(k_0)/u(c_1) = w(J(k_1)/u(c_1)).$$

Now, $k_0 = \theta^{-1}k_1$ and J is degree-γ homogeneous, so $J(k_0)/u(c_1) = \theta^{-\gamma}J(k_1)/u(c_1)$. Hence, the Bellman equation can be rewritten as

$$\theta^{-\gamma}z_1 = w(z_1),$$

which is our second equation. Total differentiation of the two equations – and elimination of $\mathrm{d}z_1$ – gives the elasticity of θ with respect to β as

$$\frac{\mathrm{d}\theta/\theta}{\mathrm{d}\beta/\beta} = \left[(1-\gamma) - \gamma\frac{zw''(z_1)/w'(z_1)}{1-(z_1w'(z_1)/w(z_1))}\right]^{-1}.$$

In the additive case, where w'' is zero, the elasticity is simply $(1-\gamma)^{-1}$. The essential point, of course, is not that the two elasticities will be different numbers, but rather, that in one case the elasticity is constant, while in the other case it may vary. In the nonadditive recursive case, the elasticity can differ as it is evaluated at different values of β. Equal percentage changes in the productivity of capital – and one may wish to view β here as the privately perceived after-tax rate of return – can have different percentage effects on the economy's long-run growth rate depending on where the economy is operating at the time the perturbation takes place. This, of course, seems quite a natural result – in contrast to the constant elasticity implied by additively separable utility.

More models of this sort – which obtain endogenous growth from some form of constant returns to scale in a convex environment – can be found in King and Rebelo (1990) and Rebelo (1991).

4. Conclusion

In this paper, I have shown how balanced-growth-consistent recursive utility functions can be derived from a rather straightforward homogeneity restriction on the aggregator. In particular, the Theorem shows how one can in practice guarantee homogeneity of the recursive utility function, and hence guarantee constancy of marginal rates of substitution along balanced paths of consumption. A loose 'corollary' to the result is that $W(x, y) = u(x)w(y/u(x))$, with u degree-γ homogeneous, constitutes a family of balanced-growth-consistent aggregators as w and u range over a feasible set of choices. This feasible set is defined by requirements such as the desirability of current consumption and future utility, bounded time perspective, continuity, as well as technical requirements guaranteeing the existence of the recursive utility function.

I have also attempted to show that such preferences can be incorporated into the basic Ramsey model of optimal capital accumulation without much additional complication beyond that which arises when utility is additive – at least for the basic existence questions. The simple analogues to the Brock–Gale and Jones–Manuelli conditions to which these preferences give rise will hopefully facilitate their use in applications.

Obviously, though, much work remains to be done. Existence results of the type given in Section 3 and in the Appendix, while perhaps interesting in themselves, are but a starting point for real dynamic modelling. There, tractability, in particular the availability of useful comparative dynamics results, is crucial. Whether these preferences will yield such results remains to be seen.

Appendix

This section contains the proofs of Proposition 1 and Theorems 1. I also describe precisely what the restrictions guaranteeing existence of the recursive utility function are, and how it is that they reduce to $\beta^\gamma\bar{\delta} < 1$ for our balanced-growth-consistent aggregators.

A.1. Proof of Proposition 1

First, note that if a utility function is recursive, so that it has an aggregator representation, and homogeneous of some degree γ, then the aggregator must satisfy $W(\lambda x, \lambda^\gamma y) = \lambda^\gamma W(x, y)$ for all (x, y) and all $\lambda \geq 0$. This follows from the fact that $U(\lambda C) = W(\lambda c_1, U(\lambda c_2, \lambda c_3, \dots)) = W(\lambda c_1, \lambda^\gamma U(c_2, c_3, \dots))$ and $U(\lambda C) = \lambda^\gamma U(C) = \lambda^\gamma W(c_1, U(c_2, c_3, \dots))$.

Given that the U is degree-γ homogeneous and W is homogeneous in the manner described above, it's straightforward to show that with such an aggregator and utility function, if W is also once-differentiable, the partial derivative $W_1(c_1, U(c_2, c_3, \dots))$ is homogeneous of degree $\gamma - 1$ in $(c_1, c_2, \dots)$ and $W_2(c_1, U(c_2, c_3, \dots))$ is homogeneous of degree zero in $(c_1, c_2, c_3, \dots)$. This is true for the case of multiple consumption goods as well as the case of a single consumption good. To see how this works, consider W_1 for the single-consumption-good case. We have

$$
\begin{aligned}
W_1(\lambda c_1, U(\lambda c_2, \dots)) &= \lim_{\varepsilon \to 0} \frac{W(\lambda c_1 + \varepsilon, \lambda^\gamma U(c_2, \dots)) - W(\lambda c_1, \lambda^\gamma U(c_2, \dots))}{\varepsilon} \\
&= \lambda^\gamma \lim_{\varepsilon \to 0} \frac{W(c_1 + \varepsilon/\lambda, U(c_2, \dots)) - W(c_1, U(c_2, \dots))}{\varepsilon} \\
&= \lambda^\gamma W_1(c_1, U(c_2, \dots))(1/\lambda) \\
&= \lambda^{\gamma - 1} W_1(c_1, U(c_2, \dots)).
\end{aligned}
$$

The consequence of these inherited homogeneity properties of W_1 and W_2 is that along a path of consumption with a constant growth rate θ, W_1 has a constant growth rate $\theta^{\gamma-1}$ and W_2 is simply constant. Hence the marginal rate of substitution is constant along a balanced path and is given by

$$MRS_{t,t+1}(C_{\mathrm{balanced}}) = \theta^{1-\gamma}/W_2.$$

A.2. Proof of Theorem 1

To understand both the proof of Theorem 1 and the subsequent section on restrictions constraining the choice of w and u, it's essential to expose at least some of the structure of the Boyd result on which Theorem 1 relies.

The notation is as in Becker and Boyd (1993). Let $X \times Y$ denote the domain of W, where X is a subset of R^m_+ and Y is a subset of R. Boyd considers the one-good case – where $X \subset R_+$ – but the result clearly generalizes. Because of the recursivity, the range of W must necessarily be a subset of Y as well. Let $A = A_1 \times A_2 \times \cdots \subset (R^\infty)^m$ such that $A_t \subset X$ for every t. Assume the topology on A is such that the shift operator S, defined by $S(c_1, c_2, \dots) = (c_2, c_3, \dots)$, and the projection operator π, defined by $\pi C = c_1$, are both continuous. Any topology at least as strong as the product topology will do. The space A is the collection of all possible consumption streams C. This space will typically be determined by the model in which the preferences are situated. For example, in the one-good case, if the model admits a maximal one-period growth rate of consumption – call it β – then A can be taken to be the collection of all sequences C which are dominated by $\lambda \times (1, \beta, \beta^2, \dots)$ for some $\lambda > 0$. In the language of Riesz spaces, this A would be the Riesz ideal generated by the sequence $(1, \beta, \beta^2, \dots)$. In most capital accumulation contexts, A can be taken to be the Riesz ideal generated by the path of pure accumulation. If the model is such that there is a maximum sustainable level of consumption $\bar{c}$, then we may take $A_t = [0, \bar{c}]$ for all t.

With this notation in hand, what the 'Continuous Existence Theorem' demonstrates is that U exists as the fixed point of a strict contraction when, in addition to $W_2 > 0$ and $\sup W_2(x, y) \equiv \bar{\delta} < +\infty$, the aggregator satisfies:

Continuity. $W(x, y)$ is continuous in (x, y).

ϕ-Boundedness. There is a positive function ϕ, continuous on A, such that $\sup\{|W(\pi C, 0)|/\phi(C)\colon C \in A\}$ is finite.

Contractiveness. Recall from above that $\bar{\delta}$ was defined as the least upper bound of W_2. We then require $\bar{\delta}\sup\{\phi(SC)/\phi(C)\colon C \in A\} < 1$, where ϕ is the same function from the previous assumption.

Becker and Boyd denote the set of continuous, ϕ-bounded functions from A to R by C_ϕ, which is a Banach space when endowed with the norm $\|\cdot\|_\phi$ defined by

$$\|U\|_\phi = \sup\{|U(C)|/\phi(C)\}.$$

It is on this space which the operator T_W of Theorem 1 lives.

The proof of the theorem is in two simple steps. First, I show that, under the assumptions on W, the operator T_W takes degree-γ homogeneous functions into degree-γ homogeneous functions. Second, I show that the (pointwise) limit of a sequence of degree-γ homogeneous functions is a degree-γ homogeneous function. Since the iterates of T_W converge ϕ-uniformly to a unique utility function U from any initial point in C_ϕ, U must be degree-γ homogeneous.

Suppose that $\xi \in C_\phi$ and that ξ is degree-γ homogeneous. Then, for any C and $\lambda > 0$, we have

$$\begin{aligned}(T_W\xi)(\lambda C) &= W(\lambda c_1, \xi(\lambda c_2, \ldots)) = W(\lambda c_1, \lambda^\gamma \xi(c_2, \ldots))\\ &= \lambda^\gamma W(c_1, \xi(c_2, \ldots)) = \lambda^\gamma (T_W\xi)(C),\end{aligned}$$

which is just degree-γ homogeneity for $T_W\xi$.

The second step is fairly trivial. Suppose $\{f^n\}_{n=1}^{\infty}$ is a sequence of degree-γ homogeneous functions which converges pointwise to a limit f. For any x in the domain of the f^n's and f and any $\lambda > 0$, consider $|f(\lambda x) - \lambda^\gamma f(x)|$. Can this be nonzero? The answer is no, for consider:

$$\begin{aligned}|f(\lambda x) - \lambda^\gamma f(x)| &= |f(\lambda x) - f^n(\lambda x) + f^n(\lambda x) - \lambda^\gamma f(x)|\\ &\leq |f(\lambda x) - f^n(\lambda x)| + |f^n(\lambda x) - \lambda f(x)|\\ &= |f(\lambda x) - f^n(\lambda x)| + \lambda^\gamma |f^n(x) - f(x)|,\end{aligned}$$

where the last equality comes about because the f^n are assumed degree-γ homogeneous. Because the f^n converge pointwise to f, the last terms on the right-hand side can be made arbitrarily small by a sufficiently large choice of n. Thus, we must have $f(\lambda x) = \lambda^\gamma f(x)$. Since x and λ were arbitrary, f is degree-γ homogeneous.

A.3. Existence restrictions on w and u

In order to see what we must require of w and u to guarantee existence of the recursive utility function, we need to consider the three assumptions listed in the last section.

What the continuity assumption demands is clear. As for the other two conditions, note first that $W(\pi C, 0) = u(c_1)w(0)$. For ϕ-boundedness then, it's enough to have a ϕ of the form $\phi(C) = 1 + u(c_1)$ – since $u(c_1)w(0)/(1 + u(c_1))$ is bounded above and below for all c_1. As for the contraction property, this really only makes sense relative to the space A of potential consumption streams, which in turn will be model-dependent. For concreteness, suppose that the model in which the preferences are situated is such that there is a maximal long-run one-period growth rate of consumption, β – i.e., there is a β such that $C \in A$ implies $\max_i\{c_{it}\} \le \lambda\beta^t$ for some $\lambda > 0$. Again using $\phi(C) = 1 + u(c_1)$ the contraction property is satisfied if $\bar{\delta}\beta^\gamma < 1$.

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